



# Three-loop HTLpt thermodynamics at finite temperature and chemical potential: Resummation versus lattice data

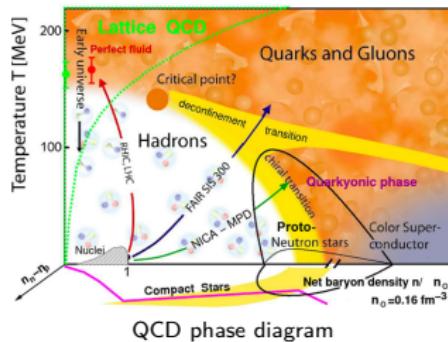
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2. References: JHEP 1405 (2014) 027 and JHEP 1312 (2013) 055

# Introduction

1. QCD equation of state - early universe, heavy-ion collisions, and compact stars



2. Quark-gluon plasma strongly coupled for  $T$  close to  $T_c$ . Weakly interacting for higher temperatures? Quasiparticle picture? Resummation?
3. Lattice. Finite baryon chemical potential? (See also Vuorinen's talk)

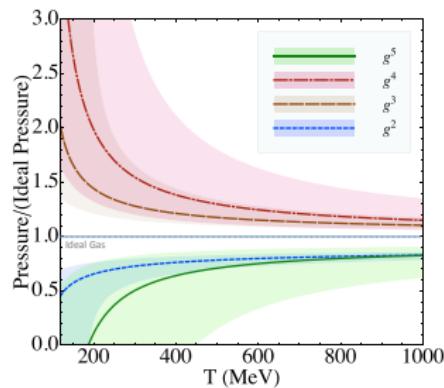
## Weak-coupling expansion

1. Weak-coupling expansion of the thermodynamic functions has a long history.  $\mathcal{F}$  is known to order  $g^6 \log g$ .<sup>1</sup>
2. Expansion poorly convergent (generic) and seems to be associated with the soft scale  $gT$ .
3. Goal: gauge-invariant framework with better convergence properties+able to describe dynamical properties+easy to generalize to finite  $\mu_q$ .

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<sup>1</sup> Arnold and Zhai, Braaten and Nieto, Kastening '94/'95, Kajantie, Laine, Rummukainen, and Schröder '02, Vuorinen '03.

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Weak-coupling expansion of the normalized pressure in three-flavor massless QCD as a function of the temperature  $T$  for zero  $\mu_B$ .

## Screened perturbation theory

## 1. Massless scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{g^2}{24}\phi^4.$$

$$= \frac{-1}{\dots} + \frac{1}{\dots}$$

## 2. Infrared divergences at higher orders

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## 3. Reorganization of the perturbative series <sup>2</sup>

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2, \\ \mathcal{L}_{\text{int}} &= -\frac{1}{2}m^2\phi^2 + \frac{g^2}{24}\phi^4.\end{aligned}$$

## 4. Expansion about an ideal gas of massive quasiparticles

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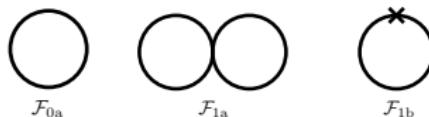
<sup>2</sup>F. Karsch, A. Patkos, and P. Petreczky '97

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## 5. Feynman rules:

$$\frac{1}{P^2 + m^2} \quad \text{---} \quad g^2 \quad \times \quad -m^2 \quad \text{---} \quad \bullet \quad \text{---}$$

## 6. Vacuum diagrams



## 7. Calculate vacuum diagrams to desired order and finally give a prescription for the mass parameter $m$ .

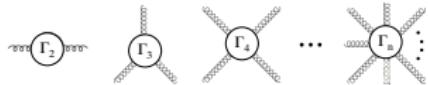
## Hard-thermal-loop perturbation theory

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{Tr}[G_{\mu\nu}G^{\mu\nu}] + i\bar{\psi}\gamma^\mu D_\mu\psi + \mathcal{L}_{\text{gh}} + \mathcal{L}_{\text{gf}} + \Delta\mathcal{L}_{\text{QCD}}.$$

1. For soft momenta, one needs effective propagators <sup>3</sup>

$$\text{wavy line with two gluon lines} = \left( \text{bare propagator} + \text{loop correction} \right) g^2 T^2$$

2. Dressed  $n$ -point vertices



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<sup>3</sup>Braaten and Pisarski '90

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3. Expansion point is gas of massive quasiparticles by adding HTL Lagrangian <sup>4</sup>

$$\mathcal{L}_{\text{HTL}} = -\frac{1}{2}(1-\delta)m_D^2 \text{Tr} \left( G_{\mu\alpha} \left\langle \frac{y^\alpha y_\beta}{(y \cdot D)^2} \right\rangle_{\hat{y}} G^{\mu\beta} \right) + (1-\delta)i m_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{y_\mu}{y \cdot D} \right\rangle_{\hat{y}} \psi ,$$

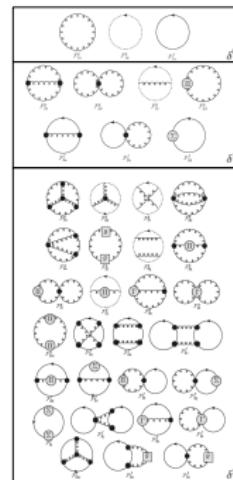
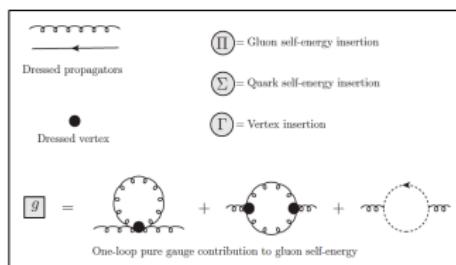
4.  $\delta$  is formal HTLpt expansion parameter.  $\delta = 1$  at the end.
5. Expansion generates effective propagators and vertices.
6. Prescription for  $m_D$  and  $m_q$ .

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<sup>4</sup>Braaten and Pisarski 1990

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## Feynman rules and vacuum diagrams



# Three-loop HTLpt thermodynamics at finite temperature and chemical potential: Resummation versus lattice data

Final result

$$\begin{aligned}
 \frac{\Omega_{\text{NNLO}}}{\Omega_0} = & \frac{7}{4} \frac{d_F}{d_A} \frac{1}{N_f} \sum_f \left( 1 + \frac{120}{7} \hat{\mu}_f^2 + \frac{240}{7} \hat{\mu}_f^4 \right) - \frac{s_F \alpha_s}{\pi} \frac{1}{N_f} \sum_f \left[ \frac{5}{8} \left( 1 + 12 \hat{\mu}_f^2 \right) \left( 5 + 12 \hat{\mu}_f^2 \right) \right. \\
 & - \frac{15}{2} \left( 1 + 12 \hat{\mu}_f^2 \right) \hat{m}_D - \frac{15}{2} \left( 2 \ln \frac{\hat{\Lambda}}{2} - 1 - \aleph(z_f) \right) \hat{m}_D^3 + 90 \hat{m}_q^2 \hat{m}_D \Big] \\
 & + \frac{s_{2F}}{N_f} \left( \frac{\alpha_s}{\pi} \right)^2 \sum_f \left[ \frac{15}{64} \left\{ 35 - 32 \left( 1 - 12 \hat{\mu}_f^2 \right) \frac{\zeta'(-1)}{\zeta(-1)} + 472 \hat{\mu}_f^2 + 1328 \hat{\mu}_f^4 \right. \right. \\
 & + 64 \left( - 36 i \hat{\mu}_f \aleph(2, z_f) + 6 \left( 1 + 8 \hat{\mu}_f^2 \right) \aleph(1, z_f) + 3 i \hat{\mu}_f \left( 1 + 4 \hat{\mu}_f^2 \right) \aleph(0, z_f) \right) \Big\} \\
 & \left. - \frac{45}{2} \hat{m}_D \left( 1 + 12 \hat{\mu}_f^2 \right) \right]
 \end{aligned}$$

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$$\begin{aligned}
& + \left( \frac{s_F \alpha_s}{\pi} \right)^2 \frac{1}{N_f} \sum_f \frac{5}{16} \left[ 96 \left( 1 + 12 \hat{\mu}_f^2 \right) \frac{\hat{m}_q^2}{\hat{m}_D} + \frac{4}{3} \left( 1 + 12 \hat{\mu}_f^2 \right) \left( 5 + 12 \hat{\mu}_f^2 \right) \ln \frac{\Lambda}{2} \right. \\
& + \frac{1}{3} + 4\gamma_E + 8(7 + 12\gamma_E)\hat{\mu}_f^2 + 112\hat{\mu}_f^4 - \frac{64}{15} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{32}{3} (1 + 12\hat{\mu}_f^2) \frac{\zeta'(-1)}{\zeta(-1)} \\
& \left. - 96 \left\{ 8\aleph(3, z_f) + 12i\hat{\mu}_f \aleph(2, z_f) - 2(1 + 2\hat{\mu}_f^2)\aleph(1, z_f) - i\hat{\mu}_f \aleph(0, z_f) \right\} \right] \\
& + \left( \frac{s_F \alpha_s}{\pi} \right)^2 \frac{1}{N_f^2} \sum_{f,g} \left[ \frac{5}{4\hat{m}_D} \left( 1 + 12 \hat{\mu}_f^2 \right) \left( 1 + 12 \hat{\mu}_g^2 \right) + 90 \left\{ 2(1 + \gamma_E) \hat{\mu}_f^2 \hat{\mu}_g^2 \right. \right. \\
& - \left\{ \aleph(3, z_f + z_g) + \aleph(3, z_f + z_g^*) + 4i\hat{\mu}_f [\aleph(2, z_f + z_g) + \aleph(2, z_f + z_g^*)] - 4\hat{\mu}_g^2 \aleph(1, z_f) \right. \\
& \left. \left. - (\hat{\mu}_f + \hat{\mu}_g)^2 \aleph(1, z_f + z_g) - (\hat{\mu}_f - \hat{\mu}_g)^2 \aleph(1, z_f + z_g^*) - 4i\hat{\mu}_f \hat{\mu}_g^2 \aleph(0, z_f) \right\} \right]
\end{aligned}$$

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$$\begin{aligned}
& - \frac{15}{2} \left( 1 + 12\hat{\mu}_f^2 \right) \left( 2\mathbb{L} - 1 - \aleph(z_f) \right) \hat{m}_D \Big] \\
& + \left( \frac{c_A \alpha_s}{3\pi} \right) \left( \frac{s_F \alpha_s}{\pi N_f} \right) \sum_f \left[ \frac{15}{2\hat{m}_D} \left( 1 + 12\hat{\mu}_f^2 \right) - \frac{235}{16} \left\{ \left( 1 + \frac{792}{47} \hat{\mu}_f^2 + \frac{1584}{47} \hat{\mu}_f^4 \right) \ln \frac{\hat{\Lambda}}{2} \right. \right. \\
& - \frac{144}{47} \left( 1 + 12\hat{\mu}_f^2 \right) \ln \hat{m}_D + \frac{319}{940} \left( 1 + \frac{2040}{319} \hat{\mu}_f^2 + \frac{38640}{319} \hat{\mu}_f^4 \right) - \frac{24\gamma_E}{47} \left( 1 + 12\hat{\mu}_f^2 \right) \\
& \left. \left. - \frac{44}{47} \left( 1 + \frac{156}{11} \hat{\mu}_f^2 \right) \frac{\zeta'(-1)}{\zeta(-1)} - \frac{268}{235} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{72}{47} \left[ 4i\hat{\mu}_f \aleph(0, z_f) + \left( 5 - 92\hat{\mu}_f^2 \right) \aleph(1, z_f) \right. \right. \right. \\
& \left. \left. \left. + 144i\hat{\mu}_f \aleph(2, z_f) + 52\aleph(3, z_f) \right] \right\} + 90 \frac{\hat{m}_q^2}{\hat{m}_D} + \frac{315}{4} \left\{ \left( 1 + \frac{132}{7} \hat{\mu}_f^2 \right) \mathbb{L} \right. \\
& \left. + \frac{11}{7} \left( 1 + 12\hat{\mu}_f^2 \right) \gamma_E + \frac{9}{14} \left( 1 + \frac{132}{9} \hat{\mu}_f^2 \right) + \frac{2}{7} \aleph(z_f) \right\} \hat{m}_D \Big] + \frac{\Omega_{\text{NNLO}}^{\text{YM}}}{\Omega_0}, \tag{1}
\end{aligned}$$

## Dimensional reduction

1. Well separated mass scales at weak coupling - fermions decouple

$$\Delta(P_0, p) = \frac{1}{P_0^2 + p^2}, P_0 = 2n\pi T, P_0 = (2n+1)\pi T,$$

2. Three momentum scales, hard scale  $T$ , soft scale  $gT$ , and supersoft scale  $g^2T$ .
3. Integrate out hard scale  $T$  perturbatively to obtain effective three-dimensional theory (EQCD).<sup>5</sup>



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<sup>5</sup>Braaten and Nieto '96, Kajantie, Laine, Rummukainen, and Shaposhnikov '96.

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$$\mathcal{L}_{\text{EQCD}} = \frac{1}{2} \text{Tr} [G_{ij}]^2 + \text{Tr} [(D_i A_0)^2] + M_E^2 \text{Tr} [A_0^2] + \lambda_E \text{Tr} [A_0^4] + \delta \mathcal{L},$$

4. Parameters depend on  $T, \mu, g\dots$
5. Integrate out scale  $gT$  ( $A_0$ )

$$\mathcal{L}_{\text{MQCD}} = \frac{1}{2} \text{Tr} [G_{ij}]^2 + \dots$$

Infrared divergent in perturbation theory - must use lattice simulations. Contributes first at order  $g^6$ .

## Results

1. Use one-loop running with  $\alpha_s(1.5\text{GeV}) = 0.326$ <sup>6</sup>
2. Mass prescription: use  $m_f = 0$  and  $m_D$  from EQCD<sup>7</sup>

$$\begin{aligned} \hat{m}_D^2 &= \frac{\alpha_s}{3\pi} \left\{ c_A + \frac{c_A^2 \alpha_s}{12\pi} \left( 5 + 22\gamma_E + 22 \ln \frac{\hat{\Lambda}_g}{2} \right) \right. \\ &\quad + \frac{1}{N_f} \sum_f \left[ s_F \left( 1 + 12\hat{\mu}_f^2 \right) + \frac{c_A s_F \alpha_s}{12\pi} \left( \left( 9 + 132\hat{\mu}_f^2 \right) \right. \right. \\ &\quad \left. \left. + 22 \left( 1 + 12\hat{\mu}_f^2 \right) \gamma_E + 2 \left( 7 + 132\hat{\mu}_f^2 \right) L + 4N(z_f) \right) \right. \\ &\quad \left. + \frac{s_F^2 \alpha_s}{3\pi} \left( 1 + 12\hat{\mu}_f^2 \right) \left( 1 - 2L + N(z_f) \right) - \frac{3}{2} \frac{s_F \alpha_s}{\pi} \left( 1 + 12\hat{\mu}_f^2 \right) \right] \right\}. \end{aligned}$$

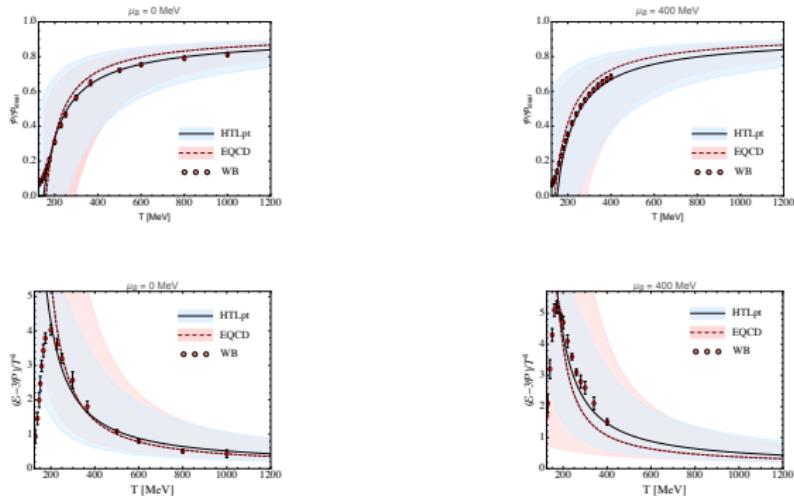
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<sup>6</sup>Bazavov '12

<sup>7</sup>Vuorinen '03

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## 3. Pressure and trace anomaly<sup>8</sup>



<sup>8</sup>Borsanyi et al '10 and '12

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## 4. Expansion of pressure

$$\frac{\mathcal{P}}{T^4} = \frac{\mathcal{P}_0}{T^4} + \sum_{ijk} \frac{1}{i!j!k!} \chi_{ijk} \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k$$

Quark susceptibilities

$$\chi_{ijk\dots} = \left. \frac{\partial^{i+j+k+\dots} \mathcal{P}(T, \mu)}{\partial \mu_u^i \partial \mu_d^j \partial \mu_s^k} \right|_{\mu_q=0}.$$

Baryon susceptibilities

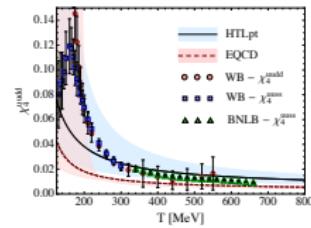
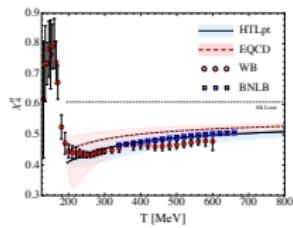
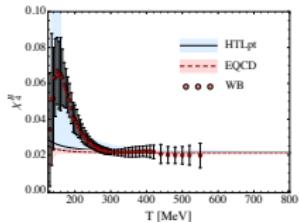
$$\chi_n^B = \left. \frac{\partial^n \mathcal{P}}{\partial \mu_B^n} \right|_{\mu_B=0}.$$

Relations

$$\chi_2^B = \frac{1}{9} [\chi_2^{uu} + \chi_2^{dd} + \chi_2^{ss} + 2\chi_2^{ud} + 2\chi_2^{us} + 2\chi_2^{ds}].$$

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## 5. Susceptibilities <sup>9</sup>



<sup>9</sup>Borsanyi et al '12, Ding et al '15

## Summary and outlook

1. Hard-thermal-loop perturbation theory represents a gauge-invariant reorganization of the perturbative series
2. Analytic result for the three-loop QCD thermodynamic potential at finite  $T$  and  $\mu$ . We also have results for  $\mu_B = 0$  and  $\mu_I \neq 0$  (lattice simulations possible!)
3. Agreement with lattice data for a number of variables is good, in particular considering that there are no fit parameters.
4. HTLpt is formulated in Minkowski space and can therefore be applied to real-time quantities as well.
5. Resummed DR also in good agreement with lattice
6. Decrease sensitivity to the renormalization scale? <sup>10</sup>

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<sup>10</sup>Kneur and Pinto '15